

# Cylindrical solutions in metric $f(R)$ gravity

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## Abstract

We study static cylindrically symmetric vacuum solutions in Weyl coordinates in the context of the metric  $f(R)$  theories of gravity. The set of the modified Einstein equations is reduced to a single equation and it is shown how one can construct exact solutions corresponding to different  $f(R)$  models. In particular the family of solutions with constant Ricci scalar ( $R = R_0$ ) is found explicitly which, as a special case ( $R = 0$ ), includes the exterior spacetime of a cosmic string. Another new solution with constant, non-zero Ricci scalar is obtained and its possible relation to the Linet-Tian solution in general relativity is discussed.

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## I. INTRODUCTION

As pointed out nicely by Weinberg in his seminal paper [1] on the cosmological constant problem, "*Physics thrives on crisis*". Perhaps the biggest crisis of the 20-th century physics which was carried over to the 21-st century is the so called cosmological constant or dark energy problem: the 120 orders of magnitude difference between the observational and theoretical values of the vacuum energy density. To overcome this crisis, different proposals have been put forward with the hope of obtaining a consistent theoretical background to the recent observation of an expanding universe which is seemingly not accessible through the standard model of cosmology. Obviously one might trace back this lack of a proper theoretical explanation to the basic ingredients of the standard model of cosmology, one of which being the Einstein field equations derived from the Einstein-Hilbert action. *Modified or alternative theories of gravity* is the paradigm under which all those theories which differ from the Einsteinian gravity are studied. One of these non-Einsteinian theories, that arose a lot of enthusiasm recently, is the so called  $f(R)$  gravity in which a function,  $f(R)$ , replaces the Einstein-Hilbert (gravitational) Lagrangian  $R$ . It seems that  $f(R)$  actions were first contemplated by Eddington [2] and later on rigorously studied by Buchdahl [3] in the context of nonsingular oscillating cosmologies. These theories could be thought of as a special kind of higher derivative gravitational theories. Having shown that these models are equivalent to scalar-tensor models of gravity it is obvious that one should in the first place check their consistency with the solar system tests of Einstein gravity [4]. In most of the models it is not possible to satisfy these tests and at the same time to account for the accelerated expansion of the Universe without bringing in new degrees of freedom. Recently it was shown that there are models of  $f(R)$  gravity in which one could account for both the accelerated expansion of the universe and the solar system tests without introducing the cosmological constant [5]-[6]. This is why spherically symmetric solutions are the most widely studied exact solutions in the context of  $f(R)$  gravity. Apart from this it is interesting, at least from theoretical point of view, to consider other exact solutions of the modified Einstein equations of  $f(R)$  theory. As in the case of the Einstein-Hilbert action one could derive field equations in  $f(R)$  gravity in two different approaches, the so called *metric* and *Palatini* approaches. But in  $f(R)$  actions, unlike the Einstein-Hilbert action or its modified version (one with a cosmological constant term), the field equations obtained by the two approaches are not the same in general. In

what follows we will be interested only in metric  $f(R)$  theories of gravity in which connection is dependent on metric  $g_{\mu\nu}$  with respect to which, the action is varied. In ordinary GR there are not that many exact solutions of the field equations for a given symmetry. Being higher derivative theory it is not unexpected to find more exact solutions in  $f(R)$  gravity and this turns out to be the case for spherically symmetric solutions [7]. Since cylindrical symmetry is the next symmetry considered normally in the study of exact solutions in GR (not just for theoretical reasons but also because they might have physical realization in objects such as cosmic strings) it seems natural to extend the studies of exact solutions in  $f(R)$  theories in the same direction. Looking for solutions with a different symmetry, as a first step, we consider static cylindrically symmetric vacuum solutions of the  $f(R)$  modified Einstein equations in this letter. It is shown how one can reduce the set of equations into a single equation which could then be utilized to construct explicit solutions. For constant curvature solutions, using the general form of a cylindrically symmetric solution in Weyl coordinates we find, among possible solutions, a generalized form of a conical (zero curvature) spacetime as well as two new (non-zero curvature) solutions with one of their parameters chosen so that it is related naively to the cosmological constant (section IV). We discuss possible relation to the so called Linet-Tian (LT) solution [8]-[9] of the modified Einstein field equations which is the cylindrical analogue of the Schwarzschild-de Sitter spacetime.

## II. FIELD EQUATIONS IN $f(R)$ GRAVITY

In this section we give a brief review of the field equations in  $f(R)$  gravity. The action for  $f(R)$  gravity is given by

$$S = \int (f(R) + \mathcal{L}_m) \sqrt{-g} d^4x. \quad (1)$$

The field equation resulting from this action in the metric approach, i.e. assuming that the connection is the Levi-Civita connection and the variation is done with respect to the metric  $g_{\mu\nu}$ , is given by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}^g + \frac{T_{\mu\nu}^m}{F(R)}, \quad (2)$$

where the gravitational stress-energy tensor is

$$T_{\mu\nu}^g = \frac{1}{F(R)} \left( \frac{1}{2}g_{\mu\nu}(f(R) - RF(R)) + F(R)^{;\alpha\beta}(g_{\alpha\mu}g_{\beta\nu} - g_{\mu\nu}g_{\alpha\beta}) \right), \quad (3)$$

with  $F(R) \equiv df(R)/dR$  and  $T_{\mu\nu}^m$  the standard matter stress-energy tensor derived from the matter Lagrangian  $\mathfrak{L}_m$  in the action (1). The vacuum equations of motion, i.e. in the absence of matter, are given by,

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \square F(R) = 0 \quad (4)$$

Contraction of the above field equations gives the following relation between  $f(R)$  and its derivative  $F(R)$

$$F(R)R - 2f(R) + 3\square F(R) = 0, \quad (5)$$

which will be employed later both to simplify the field equations and to find the general form of the  $f(R)$  function.

### III. CYLINDRICALLY SYMMETRIC VACUUM SOLUTIONS

Interested in the static cylindrically symmetric solutions of the vacuum field equations (4), we start with the general form of such a metric in the cylindrical Weyl coordinates  $(t, r, \varphi, z)$  given by [10];

$$g_{\mu\nu} = \text{diag}(-e^{2k(r)-2u(r)}, e^{2k(r)-2u(r)}, w(r)^2 e^{-2u(r)}, e^{2u(r)}). \quad (6)$$

The corresponding scalar curvature is

$$R = \frac{-wu'' + wk'' - w'u' + w'' + wu'^2}{we^{2k}}, \quad (7)$$

in which  $' \equiv \frac{d}{dr}$ . Using equation (5), the modified Einstein equations become

$$FR_{\mu\nu} - \nabla_\mu \nabla_\nu F = \frac{1}{4}g_{\mu\nu}(FR - \square F(R)). \quad (8)$$

As in the spherical case, since the metric only depends on the cylindrical radial coordinate  $r$ , one can view equation (8) as a set of differential equations for functions  $F(r)$ ,  $u(r)$ ,  $k(r)$  and  $w(r)$ . In this case both sides are diagonal and hence we have four equations. Differentiating equation (5) with respect to  $r$  we have the extra, consistency relation for  $F(r)$ ,

$$RF' - R'F + 3(\square F)' = 0. \quad (9)$$

Any solution of equation (8) must satisfy this relation in order to be also a solution of the original modified Einstein's equations. From equation (8) it is found that

$$\frac{FR_{\mu\mu} - \nabla_\mu \nabla_\mu F}{g_{\mu\mu}} = \frac{1}{4}(FR - \square F(R)). \quad (10)$$

In other words the combination  $A_\mu \equiv \frac{FR_{\mu\mu} - \nabla_\mu \nabla_\mu F}{g_{\mu\mu}}$  (with fix indices) is independent of the index  $\mu$  and therefore  $A_\mu = A_\nu$  for all  $\mu, \nu$ . This allows us to write the following independent field equations;

$$-F'' + 2F'(k' - u') + F\left(-\frac{2k'w'}{w} + \frac{w''}{w} + 2u'^2\right) = 0 \quad (11)$$

$$Fw^2\left(-k'' - \frac{k'w'}{w} + \frac{w''}{w}\right) + F'(k'w^2 - ww') = 0 \quad (12)$$

$$Fw\left(-k'' + 2u'' - \frac{k'w'}{w} + \frac{2w'u'}{w}\right) + F'(wk' - 2u'w) = 0, \quad (13)$$

corresponding to  $A_t = A_r$ ,  $A_t = A_\phi$  and  $A_t = A_z$  respectively. Therefore, any set of functions  $u(r)$ ,  $k(r)$  and  $w(r)$  satisfying the above equations would be a solution of the modified Einstein field equations (8) for a given  $F(r)$  satisfying equation (5). Obviously it is not an easy task to find a general solution to the above equations, so in the following section we discuss the simple but important case of solutions with constant curvature.

#### IV. CONSTANT CURVATURE SOLUTIONS

It is known that some of the vacuum constant curvature solutions in  $f(R)$  gravity are equivalent to vacuum solutions in Einstein theory with the same symmetry. For example it is shown in [7] that in the spherically symmetric case the corresponding  $f(R)$  solutions include the Schwarzschild-de-Sitter space for a specific choice of one of the constants of integration. For cylindrical symmetry, in Einstein gravity, static vacuum solutions were found almost immediately after their spherical counter parts by Levi-Civita [11] but those with a cosmological constant have to wait another 60 years to be found independently by Linet [8] and Tian [9]. Their solution reduces to that of a cosmic string in the limit  $r \rightarrow 0$ . Looking for cylindrically symmetric solutions in  $f(R)$  gravity, here we consider the simple but physically important case of static constant curvature spacetimes. To do so, taking  $R = \text{constant}$  in the field equations (11), (12) and (13), we arrive at the following set of

equations:

$$2u'^2 + \frac{w''}{w} - 2\frac{k'w'}{w} = 0 \quad (14)$$

$$k'' + \frac{k'w'}{w} - \frac{w''}{w} = 0 \quad (15)$$

$$2u'' + 2\frac{w'u'}{w} - \frac{k'w'}{w} - k'' = 0. \quad (16)$$

From equations (15) and (16) one could obtain the following two equations:

$$u' = \frac{1}{2} \frac{w' + c_2}{w} \quad (17)$$

$$k' = \frac{w' + c_1}{w}, \quad (18)$$

in which  $c_1$  and  $c_2$  are integration constants. By substituting equations (17) and (18) in (14) we obtain the following differential equation for  $w(r)$ :

$$\frac{1}{2} \left( \frac{w' + c_2}{w} \right)^2 + \frac{w''}{w} = 2 \frac{w'(c_1 + w')}{w^2}. \quad (19)$$

One could solve the above equation by inspection and the solutions discussed below correspond to solutions with zero and non-zero values of the Ricci scalar  $R$ .

#### **Case (1) : solution with $R = 0$**

Looking at equation (19) one could obviously arrange for a solution of  $w(r)$  linear in  $r$  (i.e  $w'' = 0$ ) in which case the metric functions are given as follows;

$$u = c_3 \pm \sqrt{\frac{c_5}{c_6}} \ln(w) \quad (20)$$

$$k = c_4 + \frac{c_5}{c_6} \ln\left(\frac{w}{c_6}\right) \quad (21)$$

$$w = c_6 r + c_7. \quad (22)$$

It could be seen that this is a Ricci flat solution (i.e  $R = 0$  in (7)) in which we should identify which one of the constants  $c_3$  to  $c_7$  correspond to physical parameters of the spacetime and which ones could be absorbed into the coordinate redefinitions [16]. Substituting the above functions back into the metric form (6) we obtain

$$ds^2 = e^{-2(c_3 \pm \sqrt{\frac{c_5}{c_6}} \ln \rho)} \left( e^{2(\tilde{c}_4 + \frac{c_5}{c_6} \ln \rho)} \left( \frac{d\rho^2}{c_6} - dt^2 \right) + \rho^2 d\phi^2 \right) + e^{2(c_3 \pm \sqrt{\frac{c_5}{c_6}} \ln \rho)} dz^2, \quad (23)$$

in which  $\tilde{c}_4 = c_4 - \frac{c_5}{c_6} \ln c_6$  and  $\rho \equiv w = c_6 r$  is the new radial coordinate by setting  $c_7 = 0$  without any loss of generality. One can also show that through the following redefinitions of the constants and the coordinates

$$m = \sqrt{\frac{c_5}{c_6}} \quad (24)$$

$$\tilde{t} = e^{\tilde{c}_4 - c_3} A^{\frac{1}{m(m+1)+1}} t \quad (25)$$

$$\tilde{z} = e^{c_3} A^{\frac{\mp m}{m(m+1)+1}} z \quad (26)$$

$$\tilde{\phi} = e^{-c_3} A^{-\frac{1 \mp m}{m(m+1)+1}} \phi \quad (27)$$

$$\tilde{\rho} = A^{\frac{1}{m(m+1)+1}} \rho \quad (28)$$

$$A = e^{\frac{\tilde{c}_4 - c_3}{c_6}}, \quad (29)$$

the above metric reduces to

$$ds^2 = \tilde{\rho}^{2m(m+1)} (d\tilde{\rho}^2 - d\tilde{t}^2) + \tilde{\rho}^{2\mp 2m} d\tilde{\phi}^2 + \tilde{\rho}^{\pm 2m} d\tilde{z}^2, \quad (30)$$

and on applying the following complex transformation [10]

$$\tilde{t} \longrightarrow i\tilde{z} \quad \tilde{z} \longrightarrow i\tilde{t}, \quad (31)$$

it transforms into the following well known metric

$$ds^2 = \tilde{\rho}^{2m(m+1)} (d\tilde{\rho}^2 + d\tilde{z}^2) + \tilde{\rho}^{2\mp 2m} d\tilde{\phi}^2 - \tilde{\rho}^{\pm 2m} d\tilde{t}^2, \quad (32)$$

which is formally similar to the Levi-Civita's static cylindrically symmetric solution in GR normally written without  $\pm$  sign but with the constant  $m$  taking both positive and negative values. It should also be noted that the range of the variable  $\tilde{\phi}$  is not in general  $(0, 2\pi]$ , not even for the flat cases of  $m = 0, 1$ .

In the case of  $m = 0$  the spacetime is conical with a deficit angle corresponding to the exterior metric of a cosmic string with the following line element [12]

$$ds^2 = (d\tilde{\rho}^2 + d\tilde{z}^2) + a_0^2 \tilde{\rho}^2 d\tilde{\phi}^2 - d\tilde{t}^2, \quad (33)$$

in which  $a_0 = c_6 e^{-c_4}$  is the conical parameter related to the *gravitational mass per unit length of the spacetime*,  $\eta$ , as [13]

$$a_0 = c_6 e^{-c_4} = 1 - 4\eta, \quad (34)$$

such that  $0 < a_0 < 1$  for  $0 < c_4 < \infty$  (taking  $c_6 = 1$ ). This metric, exposing the geometry around a straight cosmic string, is locally identical to that of flat spacetime however it is not globally Euclidian since the angle  $\tilde{\phi} = a_0\phi$  varies in the range  $0 \leq \tilde{\phi} < B$  where  $B = 2\pi a_0 = 2\pi(1 - 4\eta)$ .

From equation (5) it is seen that this metric is a solution for any form of  $f(R)$  for which  $f(R = 0) = 0$ , in other words,  $f(R)$  must be a linear superposition of  $R^n$  with  $n \geq 1$  of which the commonly considered model is  $f(R) = R + \frac{R^2}{\Lambda}$  and obviously it is not a solution of the widely studied model for which  $f(R) = R - \frac{\mu^4}{R}$ .

### **Case (2) : solutions with $R = \text{constant} \neq 0$**

We discuss two solutions of this type here:

**A)**-The first solution could be obtained by noting the simplifying fact that equation (16) is satisfied for  $k = 2u$  so that equations (14) and (15) reduce to,

$$2u'^2 + \frac{w''}{w} - 4\frac{u'w'}{w} = 0 \quad (35)$$

$$2u'' + \frac{2u'w'}{w} - \frac{w''}{w} = 0, \quad (36)$$

leading to the following equation,

$$u'' + u'^2 - \frac{u'w'}{w} = 0, \quad (37)$$

and consequently to the following relation between functions  $u$  and  $w$ ,

$$e^u = D \int w d\rho, \quad (38)$$

in which  $D$  is a constant to be determined later. On the other hand using (37), from equation (7) we have,

$$R \equiv R_0 = \frac{w''}{w} e^{-4u}. \quad (39)$$

Substituting from (38) we have;

$$\frac{w''}{w} = R_0 (D \int w d\rho)^4 \quad (40)$$

. A solution to this integro-differential equation is given by;

$$w = \rho^{-3/2} \quad D = \left(\frac{15}{64R_0}\right)^{1/4} \quad (41)$$



so that the metric takes the form

$$ds^2 = 4D^2\rho^{-1}(-dt^2 + dz^2 + d\rho^2) + \frac{\rho^{-2}}{4D^2}d\phi^2, \quad (42)$$

or in a new coordinate system with  $\bar{\rho} = 4D\rho^{1/2}$

$$ds^2 = 64D^4\bar{\rho}^{-2}(-dt^2 + dz^2) + d\bar{\rho}^2 + 64D^2\bar{\rho}^{-4}d\phi^2. \quad (43)$$

Now if one takes  $R_0 = 4\Lambda$ , as in the case of the Einstein field equations in the presence of  $\Lambda$ , then the constant  $D$  introduced above is related to the cosmological constant through

$$D^4 = \frac{15}{256}\Lambda^{-1}. \quad (44)$$

Of course one should be careful with this kind of identification of the spacetime parameter as will be discussed later.

**B)**-The other solution with constant non-zero Ricci scalar could be found by starting from the definition  $p := p(w) = \frac{dw}{dr}$  in terms of which the equation (19) can be integrated to obtain the following first integral:

$$\begin{aligned} \ln(w) - \frac{1}{3} \ln(3p(w)^2 - 2p(w)c_2 - c_2^2 + 4p(w)c_1) \\ - (c_2 \frac{1}{3} \arctan(\frac{\frac{3}{2}p(w) - \frac{1}{2}c_2 + c_1}{\sqrt{-c_2^2 + c_2c_1 - c_1^2}}) + \\ c_1 \frac{2}{3} \arctan(\frac{\frac{3}{2}p(w) - \frac{1}{2}c_2 + c_1}{\sqrt{-c_2^2 + c_2c_1 - c_1^2}}))(\sqrt{-c_2^2 + c_2c_1 - c_1^2})^{-1} + C = 0. \end{aligned} \quad (45)$$

Also in terms of the same function the radial coordinate and the metric functions are given by;

$$r = \int \frac{dw}{p(w)} \quad (46)$$

$$u(w) = 1/2 \int \frac{p(w) + c_2}{wp(w)} dw \quad (47)$$

$$k(w) = \int \frac{p(w) + c_1}{wp(w)} dw. \quad (48)$$

Again it does not seem to be an easy task to find solutions for the complicated equation (45), but in principle for each set of the values for the integration constants  $c_1$ ,  $c_2$  and  $C$  we have a solution for  $p(w)$  and correspondingly solutions for metric functions  $k(r)$  and  $u(r)$ . One such solution, looking at equation (45), could be obtained by choosing  $c_2 = 2c_1$ . In this case, defining  $A \equiv e^C$ , we find the following solution for the metric functions

$$k(w) = \ln(w) - \frac{1}{\sqrt{3}} \operatorname{arctanh}\left(\frac{\sqrt{A^3w^3 + 4c_1^2}}{2c_1}\right) \quad (49)$$

$$u(w) = \frac{1}{2} \ln(w) - \frac{1}{\sqrt{3}} \operatorname{arctanh} \left( \frac{\sqrt{A^3 w^3 + 4 c_1^2}}{2 c_1} \right), \quad (50)$$

in which  $w$  is a solution to the following equation

$$3\left(\frac{dw}{dr}\right)^2 - e^{3C} w^3 - 4c_1^2 = 0. \quad (51)$$

Apart from the trivial solution  $w = (-4c_1^2)^{1/3} e^{-C}$  (which is indeed equal to zero by equation (19) for a constant  $w$  and the fact that  $c_2 = 2c_1$ ) one could show that there is a solution in terms of the Weierstrass  $\wp$  function [17] as follows;

$$w = \text{Weierstrass}\wp\left(\frac{2^{1/3}}{6}\sqrt{3}e^C r + d, 0, -4c_1^2\right)e^{-C}2^{2/3}. \quad (52)$$

The constant curvature of the spacetimes given by the functions (49), (50) and (52) could be found through equation (7) in the coordinate system  $(t, w, \phi, z)$ , for which we find  $R_0 = A^3$ . Now if again this is going to be compared with the solutions of the Einstein field equations in the presence of the cosmological constant for which  $R_0 = 4\Lambda$ , then the correspondence  $R_0 = e^{3C} = 4\Lambda$  will fix the value of the constant  $C$  as

$$C = \frac{1}{3} \ln 4\Lambda. \quad (53)$$

By looking at equation (5) it is clear that for constant Ricci scalar solutions ( $R = R_0$ ), no matter what the symmetry,  $f(R)$  should satisfy the relation  $F(R_0)R_0 = 2f(R_0)$ . Unlike the solution with  $R_0 = 0$ , the above two solutions satisfy this relation for the commonly considered model of  $f(R) = R - \frac{\mu^4}{R}$  with  $R_0^2 = 3\mu^4$  if we have

$$D^8 = \left(\frac{5}{64}\right)^2 \frac{3}{\mu^4} \quad (54)$$

$$C = \frac{1}{6} \ln(3\mu^4) \quad (55)$$

in cases **A** and **B** respectively.

## V. DISCUSSION AND SUMMARY

Studies on the exact  $f(R)$  gravity solutions are mostly restricted to the spherically symmetric case mainly due to the solvability of the equations and also more importantly the fact that one could compare the results with the solar system observations/experiments based on the schwarzschild solution as the spacetime metric around a spherical mass such as Sun.

Here we have examined cylindrically symmetric solutions in metric  $f(R)$  gravity in a general form given by (6). Restricted by the complexity of the field equations we have only examined solutions with constant Ricci scalar. In the case of Ricci flat solutions we have found a generalized form of a conical spacetime (with zero curvature) which, as a special case, includes the cosmic string spacetime. In the non-zero Ricci scalar case we have obtained two new solutions and in both of them a parameter is identified as the cosmological constant through the comparison of their Ricci scalar with that of the modified Einstein field equations (in the presence of the cosmological constant). Obviously neither of these spacetimes are asymptotically flat nor they behave regularly in the limit  $\rho \rightarrow 0$ , and so one can not use their asymptotic behaviour to compare their parameters with those solutions known in GR having the same symmetry. Therefore one should note that either of the identifications (44) and (53), is a very naive one in the sense that we have not compared the corresponding spacetimes (nor we have studied their specifications) with one already known in the context of Einstein GR. Even if there are any solutions in GR comparable to these solutions (which we are not aware of), since the field equations in  $f(R)$  gravity are in general of higher order compared to their counterparts in GR, the correct identification should be made through matchings of different patches of the whole manifold. For example in the cylindrical case these might correspond to the exterior and interior solutions of a cylindrical shell if one of the parameters is going to be interpreted one way or another as the mass of the shell [7].

Another point need to be mentioned Incidentally it should be noted that the metric (43) has the same general form as the LT solution introduced in [9] (see also [13]) [18]. Unless one could find a solution to the equation (40) which exactly corresponds to the LT solution in Einstein gravity, it seems a reasonable conjecture to say that the solution (43) is the  $f(R)$  analogue of LT solution in GR. Finally it should be noted that in the vacuum case, for  $R = \text{constant}$  the metric and the Palatini  $f(R)$  formulations are dynamically equivalent [15]. Therefore all the above solutions are also solutions in the Palatini gravity with the same  $f(R)$  [19].

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- [16] Note that now the constants  $c_1$  and  $c_2$  could be written in terms of the constants  $c_5$  and  $c_6$ .
- [17] Also known as Weierstrass elliptic function, is a doubly periodic function. It is usually written either as  $(z; g_1, g_2)$  or  $(z|\omega_1, \omega_2)$  in which  $g_1$  and  $g_2$  are called elliptic invariants and are given in terms of the function's semi-periods  $\omega_1$  and  $\omega_2$  [14].

[18] It is shown in [9] that for a cosmic string in the presence of a positive cosmological constant, one could write the metric in the following general form

$$ds^2 = \cos^{4/3}(\lambda\rho)(dt^2 - dz^2) - d\rho^2 - \lambda^{-2} \frac{\sin^2(\lambda\rho)}{\cos^{2/3}(\lambda\rho)} d\phi^2$$

in which  $\lambda = \frac{1}{2}(3\Lambda)^{1/2}$  and other parameters are set equal to 1 [13].

[19] We are grateful to M. L. Ruggiero for pointing this out to us.